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Gravitational phase transition of fermionic matter in a general-relativistic framework

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Abstract. The Thomas–Fermi model at finite temperature is extended to describe a system of self-gravitating weakly interacting massive fermions in a general-relativistic framework. The existence and properties of the gravitational phase transition in this model are investigated numerically. It is shown that when a nondegenerate gas of weakly interacting massive fermions is cooled below some critical temperature, a condensed phase emerges, consisting of quasidegenerate fermion stars. For fermion masses of 10 to 25 keV, these fermion stars may very well provide an alternative explanation for the supermassive compact dark objects that are observed at galactic centers.

1 Introduction

The ground state of a condensed cloud of fermionic matter, interacting only gravitationally and having a mass M below the Oppenheimer–Volkoff (OV) limit [1], is a cold fermion star in which the degeneracy pressure balances the gravitational attraction of the fermions. Degenerate stars of fermions in the mass range between 10 and 25 keV are particularly interesting [2], as they could explain, without resorting to the black-hole hypothesis, at least some of the features observed around the supermassive compact dark objects with masses in the range of $M = 10^{6.5}$ to $10^{9.5}$ solar masses. Those are reported to exist at the centers of a number of galaxies [3–8], including our own [9, 10], and quasistellar objects (QSO) [11–14]. Indeed, a few Schwarzschild radii away from the object, there is little difference between a supermassive black hole and a fermion star of the same mass near the OV limit [15,16].

The purpose of this paper is to study, in the framework of a general-relativistic Thomas–Fermi model, the formation of such a star that could have taken place in the early universe shortly after the nonrelativistic fermionic matter began to dominate the radiation. This system was previously studied in the Newtonian approximation [17–22], and it was shown that the canonical and grand-canonical ensembles for such a system have a nontrivial thermodynamical limit [17–19]. Under certain conditions, these systems will undergo a phase transition that is accompanied by a gravitational collapse [21,22]. The phase transition occurs uniquely in the case of the attractive gravitational interaction of neutral fermions. As the phase transition does not happen for particles obeying Bose–Einstein or Boltzmann statistics, this phenomenon is quite distinct from the usual gravitational clustering of collisionless dark-matter particles. Gravitational condensation will also take place if the fermions have an additional short-range weak interaction, as neutrinos, neutralinos, gravitinos, and other weakly interacting massive particles do.

Effects of general relativity cannot be neglected when the total mass of the system is close to the OV limit [1]. There are three main features that distinguish the general-relativistic Thomas–Fermi model from the Newtonian one: (i) the equation of state is relativistic, (ii) the temperature and chemical potential are metric-dependent local quantities, and (iii) the gravitational potential satisfies Einstein's field equations instead of Poisson's equation.

This paper is organized as follows: In Sect. 2, we briefly discuss the nonrelativistic Thomas–Fermi model at finite temperature. In Sect. 3, this model is extended within a general-relativistic framework. In Sect. 4, we discuss the solution at zero and finite temperature and, in particular, the conditions under which the first-order gravitational phase transition occurs. Conclusions are drawn in Sect. 5.

2 Thomas–Fermi model in Newtonian gravity

Consider a system of N fermions of mass m interacting only gravitationally, confined in a spherical cavity of radius R, in equilibrium at a finite temperature T. For large N, we can assume that the fermions move in a spherically symmetric mean-field potential $V(r)$ which satisfies Poisson's equation

$$\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rV) = 4\pi G m^2 n, \tag{1}$$

G being the gravitational constant. The number density of the fermions (including antifermions) n can be expressed in terms of the Fermi–Dirac distribution (in units $\hbar = c = k = 1$)

$$n(r) = g \int \frac{d^3q}{(2\pi)^3} \left[1 + \exp\left(\frac{q^2}{2mT} + \frac{V(r)}{T} - \frac{\mu}{T}\right)\right]^{-1} . \tag{2}$$

Here g denotes the combined spin-degeneracy factor of the neutral fermions and antifermions, i.e., g is 2 or 4 for Majorana or Dirac fermions, respectively. For each solution $V(r)$ of (1), the chemical potential μ is adjusted so that the constraint

$$\int d^3r n(r) = N \tag{3}$$

is satisfied. It may be shown that a particular spherically symmetric configuration $\bar{n}(r)$ will satisfy (1)–(3) if and only if it extremizes the free energy functional defined as [18]

$$F[n] = \mu[n]N - \frac{1}{2} \int d^3r n(r)V[n] - Tg \int \frac{d^3r d^3q}{(2\pi)^3} \times \ln\left(1 + \exp\left(-\frac{q^2}{2mT} - \frac{V[n]}{T} + \frac{\mu[n]}{T}\right)\right), \tag{4}$$

where $V[n]$ and $\mu[n]$ are implicit functionals of $n(r)$ through (1)–(3). For a physical solution, we have to require that the free energy be minimal, i.e.,

$$\left.\frac{\delta F}{\delta n}\right|_{\bar{n}} = 0, \quad \left.\frac{\delta^2 F}{\delta n^2}\right|_{\bar{n}} \geq 0. \tag{5}$$

The set of self-consistency equations (1)–(3), together with (5), comprises the nonrelativistic gravitational Thomas–Fermi equation.

It may be easily shown that the following scaling property holds: If the potential energy $V(r)$ is a solution to the self-consistency equations (1)–(3), then the rescaled $\tilde{V} = A^4 V(Ar)$, with $A > 0$, is also a solution with the rescaled fermion number $\tilde{N} = A^3 N$, radius $\tilde{R} = A^{-1} R$, and temperature $\tilde{T} = A^4 T$. This property, which will be referred to as *nonrelativistic scaling*, implies the existence of a thermodynamic limit of $N^{-7/3}F$, with $N^{1/3}R$ and $N^{-4/3}T$ approaching constant values for $N \to \infty$. In this limit, the Thomas–Fermi equation becomes exact [18,19].

3 Thomas–Fermi model in general relativity

Consider a self-gravitating gas consisting of N fermions of mass m in equilibrium within a sphere of radius R. We denote by p, ρ, n, and σ the pressure, energy density, particle-number density, and entropy density of the gas, respectively. The metric generated by the mass distribution is static, spherically symmetric, and asymptotically flat, i.e.,

$$ds^2 = \xi^2 dt^2 - (1 - 2\mathcal{M}/r)^{-1} dr^2 - r^2(d\theta^2 + \sin\theta d\phi^2). \tag{6}$$

Einstein's field equations are then given by

$$\frac{d\xi}{dr} = \xi \frac{\mathcal{M} + 4\pi r^3 p}{r(r - 2\mathcal{M})} , \tag{7}$$

$$\frac{d\mathcal{M}}{dr} = 4\pi r^2 \rho, \tag{8}$$

with the boundary conditions

$$\xi(R) = \left(1 - \frac{2\mathcal{M}(R)}{R}\right)^{1/2} ; \quad \mathcal{M}(0) = 0. \tag{9}$$

The equation of state may be represented in a parametric form [23]:

$$n = g \int \frac{d^3q}{(2\pi)^3} \frac{1}{1 + e^{E/\bar{T} - \bar{\mu}/\bar{T}}} , \tag{10}$$

$$\rho = g \int \frac{d^3q}{(2\pi)^3} \frac{E}{1 + e^{E/\bar{T} - \bar{\mu}/\bar{T}}} , \tag{11}$$

$$p = g\bar{T} \int \frac{d^3q}{(2\pi)^3} \ln(1 + e^{-E/\bar{T} + \bar{\mu}/\bar{T}}) , \tag{12}$$

where g denotes the spin-degeneracy factor and $E = \sqrt{m^2 + q^2}$. The quantities $\bar{T}$ and $\bar{\mu}$ are the local temperature and chemical potential, respectively. Thermodynamic and hydrostatic equilibrium in the presence of gravity implies [24,25]

$$\bar{T}(r) = \frac{T}{\xi(r)} ; \quad \bar{\mu}(r) = \frac{\mu}{\xi(r)} . \tag{13}$$

The constants T and μ are the temperature and chemical potential at infinity. Although the matter is absent at $r = \infty$, the temperature at infinity has a physical meaning: T is the redshifted temperature [26] of the black-body radiation of a gravitating object in equilibrium at finite temperature measured at infinity. As a consequence of (13), different gravitating configurations with the same temperature at infinity may have different local temperatures. Therefore, the relevant thermal equilibrium parameter is T.

Particle-number conservation yields the constraint

$$\int_0^R dr\, 4\pi r^2 (1 - 2\mathcal{M}/r)^{-1/2}\, n(r) = N. \tag{14}$$

Given the temperature at infinity T, the set of self-consistency equations (7)–(14) defines the general-relativistic Thomas–Fermi equation. One additional important requirement is that a solution to the (7)–(14) should minimize the free energy. Based on the considerations given in [27], the free energy may be written in the form

$$F = M + \mu N - \int_0^R dr\, 4\pi r^2 \xi (1 - 2\mathcal{M}/r)^{-1/2} (p + \rho), \tag{15}$$

with $M = \mathcal{M}(R)$. The theorem on extremal properties of the free energy [27] guarantees that solutions to (7)–(14) extremize the quantity F, i.e., the free-energy functional assumes either a maximum or a minimum. We have only to find out which of the solutions are maxima and discard them as unphysical.

Next we show that, in the Newtonian limit, we recover the usual Thomas–Fermi model as defined in Sect. 2. Introducing the nonrelativistic chemical potential $\mu_{\rm NR} = \mu - m$ and the approximations $\xi = {\rm e}^{\varphi} \simeq 1+\varphi$, $E \simeq m + q^2/2m$ and $\mathcal{M}/r \ll 1$, we arrive at the Thomas–Fermi self-consistency equations [21,22] in the form

$$n = \frac{\rho}{m} = g \int \frac{{\rm d}^3 q}{(2\pi)^3} \left(1 + \exp(\frac{q^2}{2mT} + \frac{m}{T}\varphi - \frac{\mu_{\rm NR}}{T})\right)^{-1}, \tag{16}$$

$$\frac{{\rm d}\varphi}{{\rm d}r} = \frac{\mathcal{M}}{r^2}\,; \quad \frac{{\rm d}\mathcal{M}}{{\rm d}r} = 4\pi r^2 \rho\,, \tag{17}$$

$$\varphi(R) = -\frac{mN}{R}\,; \quad \mathcal{M}(0) = 0, \tag{18}$$

$$\int_0^R {\rm d}r\, 4\pi r^2 n(r) = N, \tag{19}$$

which are equivalent to the set of (1)–(3). The free energy (15) in the Newtonian limit yields

$$F = mN + \mu_{\rm NR} N - \frac{1}{2}\int_0^R {\rm d}r\, 4\pi r^2 n \varphi - \int_0^R {\rm d}r\, 4\pi r^2 p\,, \tag{20}$$

with

$$p = gT \int \frac{{\rm d}^3 q}{(2\pi)^3} \ln\left(1 + \exp(-\frac{q^2}{2mT} - \frac{m}{T}\varphi + \frac{\mu_{\rm NR}}{T})\right), \tag{21}$$

which, up to a constant, equals the nonrelativistic Thomas–Fermi free energy (4).

A straightforward thermodynamic limit $N \to \infty$, as discussed by Hertel, Thirring, and Narnhofer [18,19], is not directly applicable in the general-relativistic case. First, in contrast to the Newtonian case, there exists a limiting configuration with maximal M and N (the Oppenheimer–Volkoff limit) at zero temperature, and, as we shall shortly demonstrate, also at finite temperature. Second, the scaling property of the relativistic Thomas–Fermi equation, which will be referred to as *relativistic scaling*, is quite distinct from nonrelativistic scaling. This scaling property may be formulated as follows: If the configuration $\{\xi(r), \mathcal{M}(r)\}$ is a solution to the self-consistency equations (7)–(14), then the configuration $\{\tilde{\xi} = \xi(A^{-1}r), \tilde{\mathcal{M}} = A\mathcal{M}(A^{-1}r); A > 0\}$ is also a solution with the rescaled fermion number $\tilde{N} = A^{3/2}N$, radius $\tilde{R} = AR$, asymptotic temperature $\tilde{T} = A^{-1/2}T$, and fermion mass $\tilde{m} = A^{-1/2}m$. The free energy is then rescaled as $\tilde{F} = AF$. Hence there exists a thermodynamic limit of $N^{-2/3}F$, with $N^{-2/3}R$, $N^{1/3}T$, and $N^{1/3}m$ approaching constant values when $N \to \infty$.

4 Numerical integration

In the following we use the units in which $G = 1$. We choose appropriate length, mass and fermion number scales a, b, and c, respectively, such that

$$a = b = \sqrt{\frac{2}{g}}\,\frac{1}{m^2}, \quad c = \frac{b}{m}, \tag{22}$$

or, restoring $\hbar$, c, and G, we have

$$a = \sqrt{\frac{2}{g}}\,\frac{\hbar M_{\rm Pl}}{cm^2} = 1.3185 \times 10^{10} \sqrt{\frac{2}{g}} \left(\frac{15{\rm keV}}{m}\right)^2 {\rm km}, \tag{23}$$

$$b = \sqrt{\frac{2}{g}}\,\frac{M_{\rm Pl}^3}{m^2} = 0.8929 \times 10^{10} \sqrt{\frac{2}{g}} \left(\frac{15{\rm keV}}{m}\right)^2 M_\odot \tag{24}$$

$$c = \sqrt{\frac{2}{g}}\,\frac{M_{\rm Pl}^3}{m^3} = 5.5942 \times 10^{71} \sqrt{\frac{2}{g}} \left(\frac{15{\rm keV}}{m}\right)^3 \tag{25}$$

where $M_{\rm Pl} = \sqrt{\hbar c/G}$ denotes the Planck and $M_\odot$ the solar mass.

We are looking for a solution of the Thomas–Fermi problem as a function of temperature. For numerical convenience, let us introduce a new parameter

$$\alpha = \frac{\mu}{T} \tag{26}$$

and the substitution

$$\xi = \frac{\mu}{m}(\Phi + 1)^{-1/2}. \tag{27}$$

Using this and (22), we may write (10)–(12) in the form

$$n = \frac{1}{\pi^2}\int_0^\infty {\rm d}y\, \frac{y^2}{1 + \exp[(\sqrt{(y^2+1)/(\Phi+1)} - 1)\alpha]}\,, \tag{28}$$

$$\rho = \frac{1}{\pi^2}\int_0^\infty {\rm d}y\, \frac{y^2\sqrt{y^2+1}}{1 + \exp[(\sqrt{(y^2+1)/(\Phi+1)} - 1)\alpha]}\,, \tag{29}$$

$$p = \frac{1}{3\pi^2}\int_0^\infty {\rm d}y \times \frac{y^4}{\sqrt{y^2+1}(1 + \exp[(\sqrt{(y^2+1)/(\Phi+1)} - 1)\alpha]}. \tag{30}$$

In this way, both the fermion mass and the chemical potential are eliminated from the equation of state.

The field equations (7) and (8) now read

$$\frac{{\rm d}\Phi}{{\rm d}r} = -2(\Phi+1)\frac{\mathcal{M} + 4\pi r^3 p}{r(r - 2\mathcal{M})}\,, \tag{31}$$

$$\frac{{\rm d}\mathcal{M}}{{\rm d}r} = 4\pi r^2 \rho. \tag{32}$$

To these two equations we add

$$\frac{{\rm d}\mathcal{N}}{{\rm d}r} = 4\pi r^2 (1 - 2\mathcal{M}/r)^{-1/2} n, \tag{33}$$

imposing the particle-number constraint as a condition at the boundary:

$$\mathcal{N}(R) = N. \tag{34}$$

Equations (31)–(33) should be integrated by use of the boundary conditions at the origin

$$\Phi(0) = \Phi_0 > -1\,; \quad \mathcal{M}(0) = 0\,; \quad \mathcal{N}(0) = 0. \tag{35}$$

The parameter Φ_0, which is uniquely related to the central density and pressure, will eventually be fixed by the requirement (34). For $r \geq R$, the function Φ yields the usual empty-space Schwarzschild solution

$$\Phi(r) = \frac{\mu^2}{m^2}\left(1 - \frac{2M}{r}\right)^{-1} - 1\,, \tag{36}$$

with

$$M = \mathcal{M}(R) = \int_0^R \mathrm{d}r 4\pi r^2 \rho(r). \tag{37}$$

We now show that a solution to the general-relativistic Thomas–Fermi equation exists, provided that the number of fermions is smaller than a certain number $N_{\max}$ that depends on α and R. From (29) and (30) it follows that, for any $\alpha > 0$, the equation of state $\rho(p)$ is an infinitely smooth function and $\mathrm{d}\rho/\mathrm{d}p > 0$ for $p > 0$. Then, as shown by Rendall and Schmidt [28], there exists for any value of the central density ρ_0 a unique static, spherically symmetric solution of the field equations with $\rho \to 0$ as r tends to infinity. In that limit $\mathcal{M}(r) \to \infty$ and $\mathcal{N}(r) \to \infty$, as may be easily seen by analysis of the $r \to \infty$ limit of (31) and (32). However, the enclosed mass M and the number of fermions N within a given radius R will be finite. We can then cut off the matter from R to infinity and join the interior solution onto the empty-space Schwarzschild exterior solution by making use of equation (36). This equation together with (26) fixes the chemical potential and the temperature at infinity. Furthermore, it may be shown that our equation of state obeys asymptotically at high densities a γ law, i.e., $\rho = \text{const}\ n^\gamma$ and $p = (\gamma - 1)\rho$, with $\gamma = 4/3$. In this case, as is well known [29], there exists a limiting configuration $\{\psi_\infty(r), \mathcal{M}(r)_\infty\}$ such that M and N approach nonzero values M_∞ and N_∞, respectively, as the central density ρ_0 tends to infinity. Thus, the quantity N is a continuous function of ρ_0 on the interval $0 \leq \rho_0 < \infty$, with $N = 0$ for $\rho_0 = 0$, and $N = N_\infty$ as $\rho_0 \to \infty$. The range of N depends on α and R and its upper bound may be denoted by $N_{\max}(R, \alpha)$. Thus, for given α, R and $N < N_{\max}(R, \alpha)$ the set of self-consistency equations (28)–(37) has at least one solution.

As is evident from the equation of state (10)–(12), if we do not fix the boundary and do not constrain the particle number N, the pressure (and the density) will never vanish (except perhaps at $r = \infty$), unless $T = 0$. Thus, since we fix the boundary at R and cut off the matter from R to infinity, the pressure (and the density) will have a discontinuity. This characteristic of the nonrelativistic Thomas–Fermi model in atomic physics [30] and Newtonian gravity [18,21,22] remains in general relativity also. However, the density and the pressure decrease rapidly with r, so if R is chosen sufficiently large, the pressure and the density at the boundary will be extremely small. Furthermore, the region $r > R$ is never empty in reality, so a positive pressure at the boundary is more realistic than a vanishing pressure.

The numerical procedure is now straightforward. For a fixed, arbitrarily chosen α, we first integrate (31) and (32) numerically on the interval $(0, R)$ and find solutions for various initial Φ_0. Simultaneously integrating (33), we obtain $\mathcal{N}(R)$ as a function of Φ_0. The specific value of Φ_0 is then determined such that $\mathcal{N}(R) = N$. The chemical potential μ corresponding to this particular solution is given by (36). If we now eliminate μ using (26), we finally get the parametric dependence on temperature through α.

Let us first discuss a degenerate fermion gas ($T = 0$) as a reference point that can also be compared with the well-known results by Oppenheimer and Volkoff [1]. In this case, the Fermi distribution in (28)–(30) becomes a step function that yields an elementary integral with the upper limit $y_\mathrm{F} = \sqrt{\Phi}$ related to the Fermi momentum

$$q_\mathrm{F} = m\sqrt{\Phi}. \tag{38}$$

The equation of state can be expressed in terms of elementary functions of Φ:

$$n = \frac{1}{3\pi^2}\Phi\sqrt{\Phi}, \tag{39}$$

$$\rho = \frac{1}{8\pi^2}\left[(2\Phi + 1)\sqrt{\Phi(\Phi + 1)} - \mathrm{Arsh}\sqrt{\Phi}\right], \tag{40}$$

$$p = \frac{1}{24\pi^2}\left[(2\Phi - 3)\sqrt{\Phi(\Phi + 1)} + 3\mathrm{Arsh}\sqrt{\Phi}\right]. \tag{41}$$

The radius of the star is naturally defined as the point where the density vanishes. At this point, owing to (39), $\Phi = 0$. Therefore, we integrate equations (28)–(30) starting from $r = 0$ up to the point R where $\Phi(R) = 0$. As a result, the quantities M, N, and R are obtained as functions of the parameter Φ_0, which is related to the central particle-number density through (39). In Fig. 1, we plot the mass of the star as a function of the radius R. The maximum of the curve corresponds to the Oppenheimer–Volkoff (OV) limit [1]. The limiting values are $R_\mathrm{OV} = 3.357$, $M_\mathrm{OV} = 0.38426$, and $N_\mathrm{OV} = 0.39853$, in units of a, b, and b/m respectively. The curve to the left of the maximum represents unstable configurations that curl up around the point corresponding to the infinite central density limit.

The OV limiting mass may be regarded as a stability bound on the coupling parameter for the ground state of self-gravitating fermionic matter. Historically, Chandrasekhar [31] was one of the first to discuss and approximately determine a similar bound in the context of white dwarfs. Later on, Lieb and Yau discussed the Chandrasekhar limit more rigorously [32]. They also set rigorous stability bounds on the coupling constant for a relativistic matter interacting via Coulomb forces [33].

We now turn to the study of nonzero temperature. The quantities T, N, and, R are free parameters in our model, and their range and choice are dictated by physics. The temperature T is restricted only to positive values. The number of fermions N is restricted by the OV limit. The radius R is theoretically unlimited; practically, it should not exceed the order of interstellar distances. It is known that a classical, semidegenerate, isothermal configuration has no natural boundary in contrast to the degenerate case of zero temperature, where for given N (up to the OV limit) the radius R is naturally fixed by the condition

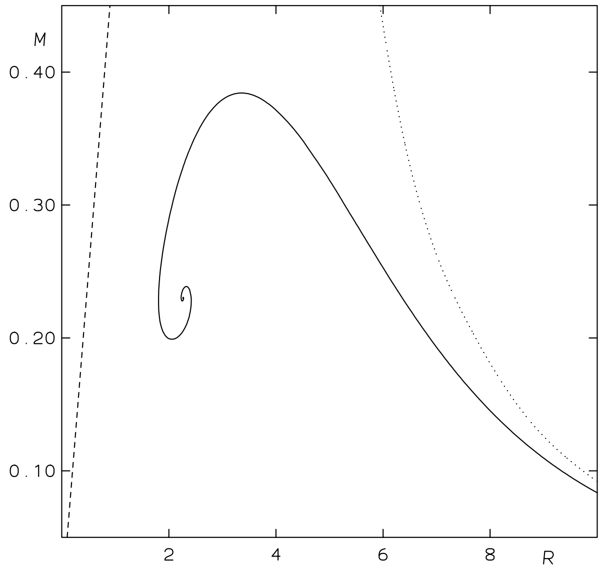

Fig. 1. Mass versus radius for fermion stars at zero temperature in the general-relativistic framework (solid line) compared with the corresponding Newtonian approximation (dotted line). The dashed line is the black hole limit $M = R/2$

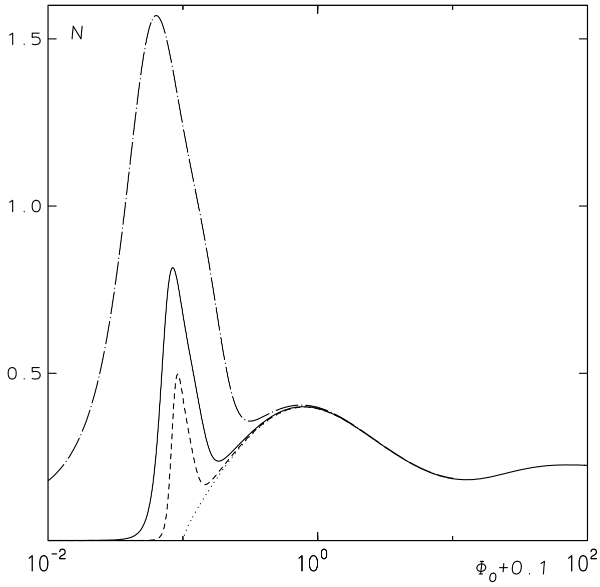

Fig. 2. Fermion number N versus central potential Φ_0 for $\alpha = \mu/T = 300$ (full line) 500 (dashed line) and 150 (dot–dashed line). T=0 is represented by dotted line

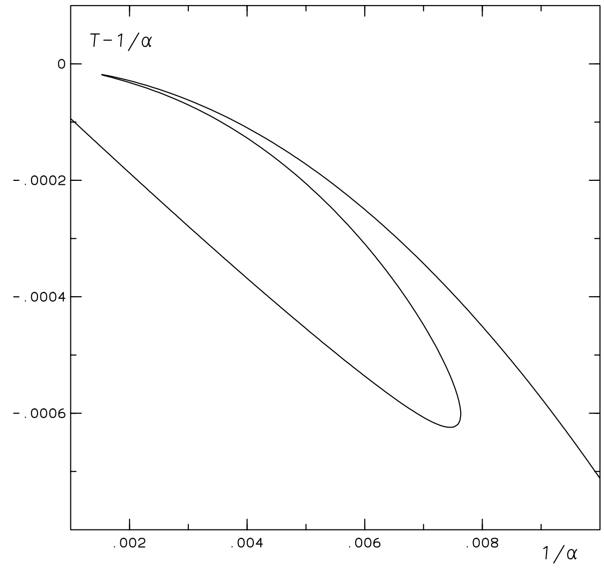

Fig. 3. Temperature T (in units of m) versus $1/\alpha$ for $N = 0.38$ and $R = 100$

of vanishing pressure and density. At nonzero temperature, if we, for example, fix only N and T and let $R \to \infty$, our gas will occupy the entire space, and hence p and ρ will vanish everywhere. If integrate the equations on the interval $(0, \infty)$ and do not restrict N, then M and N will diverge at ∞. In such a case, one has to introduce a cutoff. In the isothermal model of a similar kind by Chau, Lake, and Stone [36], a cutoff was chosen at the radius R, where the energy density was about six orders smaller than the central value. Our choice is based on the following considerations: As in the Newtonian Thomas–Fermi model [22], we expect that for a given number of fermions $N < N_{\rm OV}$, there exists a unique configuration that is a solution to the self-consistency equations (7)–(14) and which becomes a degenerate Fermi gas at $T = 0$. For such a configuration, an effective radius $R_{\rm eff} \geq R_{\rm OV}$ may be defined so that $\Phi(R_{\rm eff}) = 0$. Although the density does not vanish at this point, most of the mass will be contained inside the sphere of radius $R_{\rm eff}$. If we choose the boundary at $R \gg R_{\rm OV}$, the total mass will be dominated by the density distribution within $R_{\rm eff}$ and will be almost independent of the choice of R. Thus in the following we will work with $R = 100 \simeq 30 R_{\rm OV}$.

In Fig. 2, the fermion number N is plotted as a function of initial Φ_0 for several values of the parameter α. In contrast to the $T = 0$ case, all curves with finite α have a peak around $\Phi = 0$. The second peak corresponds to the OV limit. From this figure we can deduce that, for a given N, there is a range of α for which the Thomas–Fermi equation may have more than one solution. This is a clear indication for the existence of an instability even below the OV limit, and as a consequence, we expect that a first-order phase transition occurs.

Fixing $N = 0.38$, which is slightly below the OV limit, we can now plot the temperature as a function of α in Fig. 3. Using this figure as a parametric function for temperature, the mass, free energy, and entropy are shown as functions of temperature in Figs. 4, 5, and 6, respectively. In the temperature interval $T = (0.0015 - 0.007)m$, there are three distinct solutions of which only two are physical, namely those for which the free energy assumes a minimum. The solution that can be continuously extended to any temperature above the mentioned interval is referred to as “gas”, whereas the solution that continues to exist at low temperatures, and eventually becomes a degenerate Fermi gas at $T = 0$, will be called “condensate”. In Fig. 2,

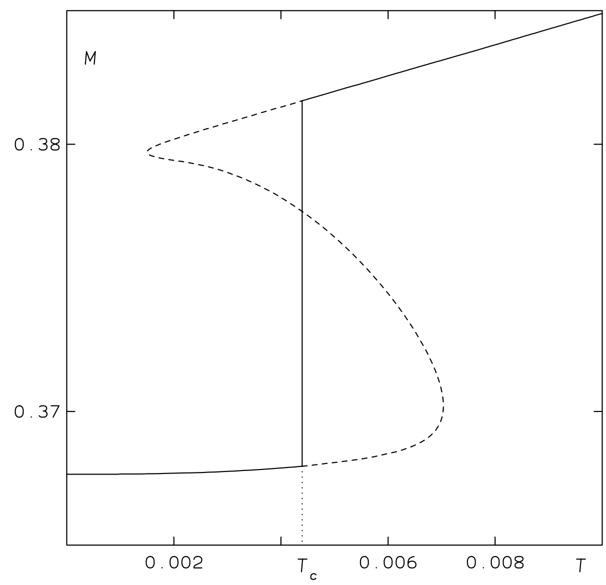

Fig. 4. Total mass (in units of b) versus temperature for $N = 0.38$ and $R = 100$

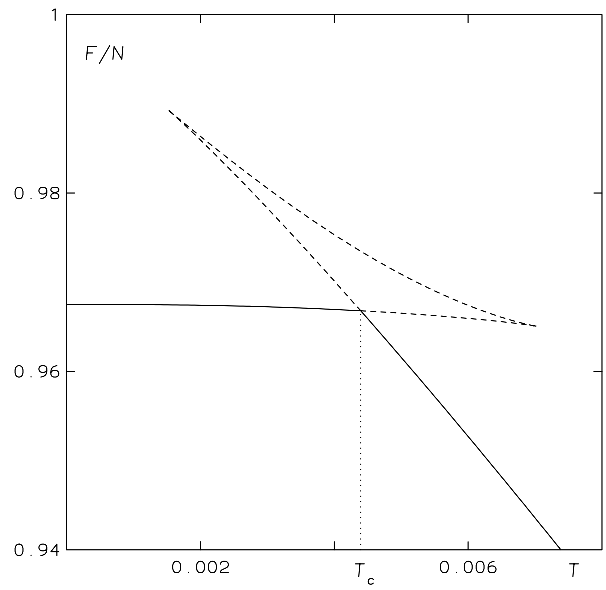

Fig. 5. Free energy per fermion F/N (in units of m) versus temperature T for $N = 0.38$ and $R = 100$

the gas is represented on each curve by the part left from the first maximum, while the part from the first minimum up to the second maximum represents the condensate. By noting that Φ_0 is negative for the gas and positive for the condensate, we may define an order parameter as

$$\delta = \Phi_0 + |\Phi_0|, \tag{42}$$

which is strictly positive in the condensed phase (ordered phase) and equal to zero in the gaseous phase (disordered phase).

The phase transition takes place at the temperature T_c, where the free energy of the gas and that of the condensate become equal. The dashed curves in Figs. 4, 5,

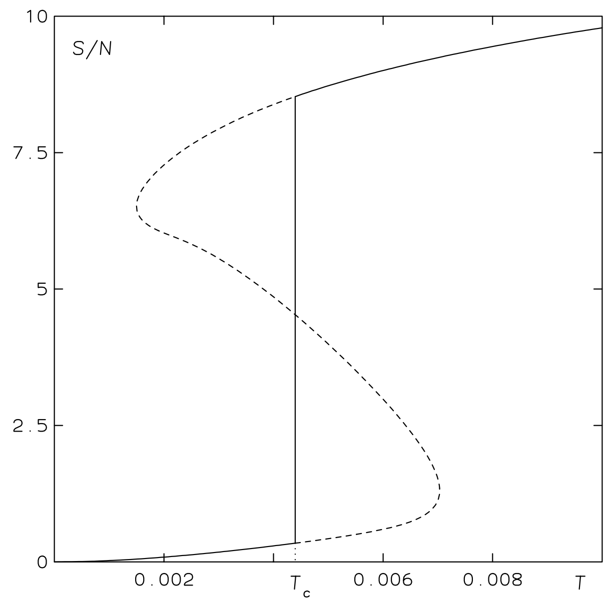

Fig. 6. Entropy per fermion S/N versus temperature T for $N = 0.38$ and $R = 100$

and 6 represent the physically unstable solution. In our example, the transition temperature is $T_c = 0.0043951m$, as indicated in the plots by the dotted line. The latent heat per fermion released during the phase transition is given by the mass difference at the point of discontinuity

$$\frac{\Delta M}{N} = 0.0438m. \tag{43}$$

So far, we have studied as an example an object with a number of fermions N just below the OV limit N_{OV}. Any object with $N < N_{\mathrm{OV}}$ will undergo a gravitational phase transition at a critical temperature (which depends on the mass, of course). With decreasing N, the cavity radius R must be appropriately increased, since the effective radius of the condensate increases, following approximately the zero-temperature mass–radius relation. As N becomes smaller, the system approaches the nonrelativistic scaling regime discussed in Sect. 2, and for $N \ll N_{\mathrm{OV}}$, the critical temperature will decrease according to

$$T_c = \mathrm{const}\, N^{4/3}, \tag{44}$$

if the cavity radius R is simultaneously rescaled as $N^{-1/3}$. In Fig. 7 we compare the critical temperature calculated in both Newtonian and general-relativistic Thomas–Fermi models, as a function of N. The nonrelativistic scaling law turns out to be very accurate for $N < 0.2\, N_{\mathrm{OV}}$.

It is important to check that the critical temperature is not very sensitive to variations of the cavity radius R for the following two reasons: First, in our model, R is arbitrary except for the requirement that it be much larger than the effective radius, which for $N = 0.38$ is of the order $R_{\mathrm{eff}} \simeq R_{\mathrm{OV}} = 3.357$. Second, if the critical temperature rapidly decreases with R, the adiabatic cooling of the gas through the universe expansion may not necessarily lead to the point of the phase transition. Figure 8

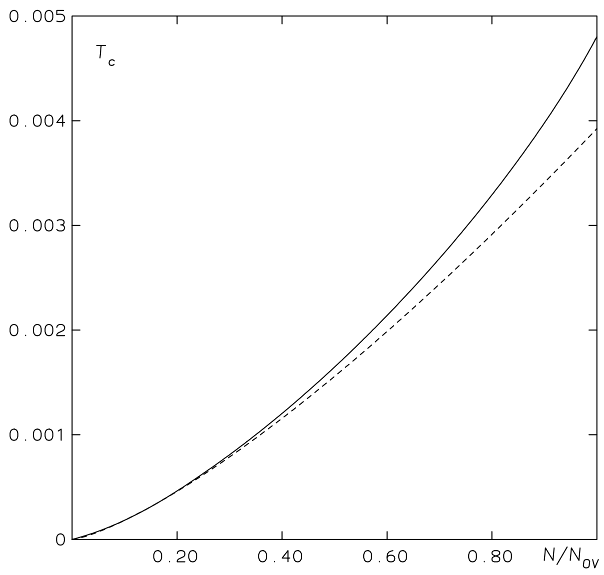

Fig. 7. Critical temperature as a function of the fermion number in the Newtonian (dashed line) and general-relativistic Thomas–Fermi model (solid line)

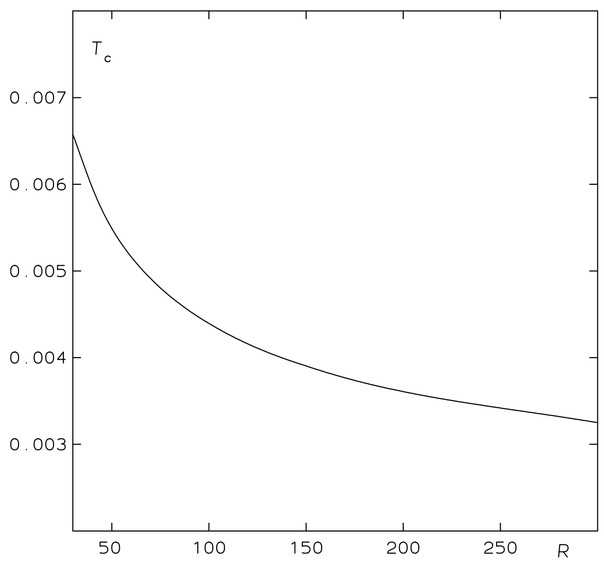

Fig. 8. Critical temperature T_c as a function of the cavity radius R for $N = 0.38$

shows that the critical temperature indeed decreases very slowly, by roughly a factor of 2 if R increases from 30 to 300. This is much weaker than the adiabatic cooling of a nonrelativistic gas, which shows an approximate $1/R^2$ dependence. Thus we conclude that the gravitational phase transition will necessarily take place in the course of the universe expansion.

5 Conclusions

In this work, we extended the Thomas–Fermi model to a general-relativistic framework. This model was then applied to a system of self-gravitating fermions. We have investigated numerically the circumstances under which this system undergoes a gravitational phase transition at nonzero temperature. This phase transition is quite distinct from the more extensively investigated strong-interaction-driven phase transition that might occur in neutron stars [34,35]. The main underlying physics here is the competition between the partial degeneracy pressure due to the Fermi–Dirac statistics and the attractive force due to the gravitational interaction. It is obvious that the application of this model to astrophysical systems will work very well if the nongravitational interactions between the individual particles can be neglected. This is certainly the case for, e.g., weakly interacting quasidegenerate heavy neutrino, neutralino, or gravitino matter [15,22,36–38], but perhaps it could be valid even for collisionless stellar systems [39,40].

Finally, let us briefly comment on a similar model by Chau, et al. [36] which was considered earlier in the context of a possible galactic massive neutrino halo. Their model differs from our approach in essentially two aspects: First, the equation of state is not consistent with the condition of thermodynamical and chemical equilibrium, i.e., with our (13) and second, the particle-number constraint (14) is not imposed in their model. Thus, in contrast to the Thomas–Fermi model discussed here, the Chau, et al. model does not describe a canonical system in equilibrium.

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